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DEHN FILLINGS THAT REDUCE THURSTON NORM

BY

ZLIL SELA[†]

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel

ABSTRACT

We construct examples for which more than one Dehn filling reduces the Thurston norm of (distinct) second homology classes of Φ_P-atoroidal 3-manifolds. We bound the number of these Dehn fillings by the number of faces of the Thurston Ball.

In this paper we investigate some questions regarding a remarkable result of D. Gabai [Ga_1, 1.7; 1.8]. This result (Theorem 1) is one of the powerful consequences of sutured manifold theory developed by Gabai. The original proof used foliations. M. Scharlemann gave a defoliated proof [Sc]. Some of the areas in knot theory to which Gabai's theorem has been applied are: super additivity of knot genus under band sum [Ga_2] [Sc], property P for band connect sum knots [T_1], and unknotting number one knots [T_2], link genus and Conway moves [Sc–T], as well as other results [W].

We start by introducing Gabai's theorem and sketching a proof of it. According to the theorem, in an important class of 3-manifolds (finite volume hyperbolic ones are just an example) the Thurston norm of a homology class remains unchanged in all but at most one Dehn filling. Under the same assumptions we show that the number of norm reducing fillings is finite, though generally greater than one. In particular we prove that in a Haken atoroidal manifold with boundary a torus, there is at most one Dehn filling per face of the unit ball for the Thurston norm that may reduce the norm of an

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embedded taut surface. We conclude by constructing manifolds for which a given (arbitrary) finite set of slopes is part of the norm reducing filling set. Throughout the paper we assume familiarity with the definitions and notations of [Sc]. Homology is always with real coefficients.

A cobordism V between two tori T_1 T_2 is called an *I-cobordism* if the mappings $h^i_*: H_1(T_i) \to H_1(V)$ induced naturally by the inclusions $h^i: T_i \to V$ are injective [Ga_1, 1.3].

Let M be a taut connected sutured manifold whose boundary is a union of tori. Let P be one of the tori boundary components. We call M *Φ_P-atoroidal* if tori parallel to P in M are the only tori I-cobordant to P.

THEOREM 1 [Ga_1, 1.8]. *Let (M, γ) be a taut connected Φ_P-atoroidal sutured manifold with boundary composed of tori. Let P be a toral component of γ, and suppose that $H_2(M, \partial M - P) \neq 0$.*

Let S be a taut surface in M, representing a homology class in $H_2(M, \partial M - P)$. Then with at most one exception (up to isotopy) S is taut in the sutured manifold obtained by filling a solid torus along P.

SKETCH OF PROOF. Let S be a taut surface in M, and let (M_1, γ_1) be the sutured manifold obtained by decomposing along S. The pieces of $R_+(\gamma_1)$, $R_-(\gamma_1)$ will clearly be isotopic to S in the original manifold M, and therefore will be incompressible and Thurston norm minimizing in $H_2(M, \partial R(\gamma_1))$. Hence (M_1, γ_1) is a taut manifold (see [Sc, 2] for the exact definitions). By [Sc, 4.19] there exists a taut hierarchy: $(M, \gamma) \to (M_1, \gamma_1) \to \cdots \to (M_n, \gamma_n)$ starting with the decomposition along S, with (M_n, γ_n) satisfying $H_2(M_n, \partial M_n - P) = 0$.

By the irreducibility of M_n and the last homology condition one can easily show that M_n is composed of balls and an I-cobordism between P and another torus T [Sc, 5.2]. By the Φ_P-atoroidal property this I-cobordism is just a product. Moreover by the construction of the taut hierarchy [Sc, 4.19] γ_n contains a non-trivial collection of simple closed curves in T.

Let $(M_n^\varphi, \gamma_n^\varphi)$ be the manifold obtained from (M_n, γ_n) after filling a solid torus along P. M_n^φ is a union of balls and one solid torus, so unless we fill a solid torus along P such that the sutures on T bound disks, $(M_n^\varphi, \gamma_n^\varphi)$ is taut, and by the pull-back lemma for taut hierarchies [Sc, 3.9], we actually have a taut hierarchy for the manifold obtained by filling a solid torus to M via φ. Specifically we get that the first decomposition is taut and hence the surface S we started with, which is isotopic to a piece of $R(\gamma_1)$, is taut as well.

DEFINITION 2. A Dehn filling for which there exists a taut surface S in M with positive Thurston norm which becomes non-taut after the filling is called a *norm-reducing filling*.

From Gabai's theorem it's not clear how many norm-reducing fillings exist, though we know there is at most one for every homology class. We show that the number of norm-reducing fillings is finite, but can be larger than one.

From our result it's clear that in general two distinct taut decompositions cannot be continued to a full taut hierarchy with the same final sutured manifold (unlike the standard Haken hierarchy). This follows because with such a continuation one can use Gabai's proof to get uniqueness of the norm-reducing filling. But according to Theorem 3 the number of norm-reducing fillings can be greater than one. The final sutured manifold structures we get may serve as an (interesting?) invariant of the original one.

Let K be the unit ball of the Thurston norm for $H_2(M, \partial M - P)$ [Th, 2], and let $n_t(M, P)$ denote the number of faces of K. ∂K is in general a $(k-1)$-dimensional cell complex, where $k = \mathrm{rank}(H_2(M, \partial M - P))$.

THEOREM 3. *Let $n_r(M, P)$ denote the number of norm-reducing fillings.*
(i) *$n_r(M, P)$ can be greater than 1 (even when $\partial M = P$).*
(ii) *$n_r(M, P) \leqq \frac{1}{2} n_t(M, P)$.*
(iii) *If $n_r(M, P) > 1$ then there exists a non-empty $(k-2)$-dimensional cell complex contained in the $(k-2)$-skeleton of ∂k which remains taut under all Dehn fillings.*

PROOF. We prove (i) by introducing the example in Fig. 1. Let T be a solid

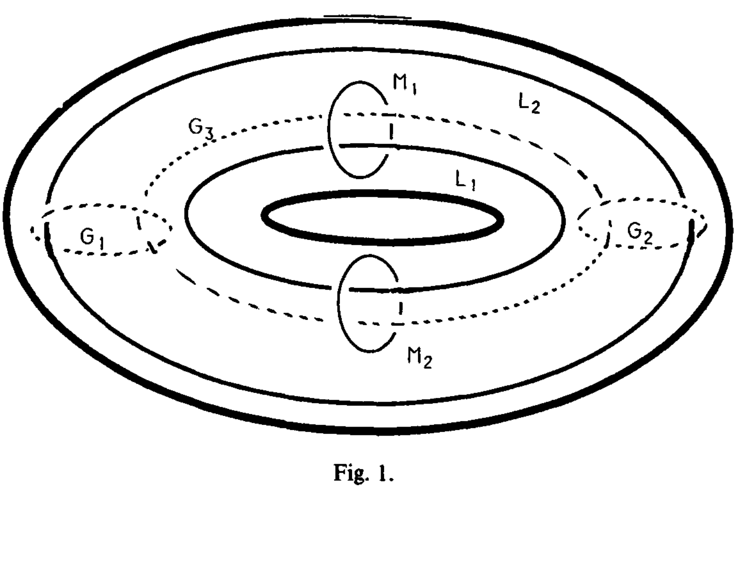

Fig. 1.

torus with boundary P. Let L_1, L_2 be the boundary of regular neighborhoods of longitudes in the interior of T, and let T' be the remnant of T after taking out these regular neighborhoods. Let M_1, M_2 be the boundary of regular neighborhoods of appropriate meridians of L_1 pushed into T'. Let T'' be the manifold one gets after taking out the last regular neighborhoods. Let G_3 be a longitude of L_1 pushed into T'', and G_1, G_2 be simple closed curves bounding disks in the original torus T once-punctured by L_2 and G_3. G_1 and G_2 would be located between M_1 and M_2 such that the picture has rotational symmetry (see diagram above). Following Thurston [Th] we do the following on each G_i, for $i = 1, 2, 3$: take out a regular neighborhood of each G_i and glue back a "high genus" (denoted by g) knot complement $C_i \cong (S^3 - \eta(k))$, such that the original meridian is identified with a longitude of k. Define M to be the resulting manifold.

CLAIM 4. Let S be a surface with boundary composed of a single longitude on each of L_1 and L_2, oriented to be anti-parallel in T, and a single meridian from each of M_1, M_2. Then $\chi_-(S) \geqq 2 + 4g$.

PROOF. S has four boundary components and because of linking number it must intersect G_1 and G_2 at least along one meridian. So for $i = 1, 2$, $\chi_-(C_i \cap S) \geqq \chi_-(\text{Seifert surface for } k) \geqq g - 1$.

Hence $x([S]) \geqq 4g$, for every class represented by one of the above surfaces.

On the other hand, it is clear that if we fill a torus along P to M such that the longitude of P becomes trivial, we can find S_∞ homologous to some surface S from above with $\chi_-(S_\infty) = 1 + (4g - 1) = 4g$.

(S_∞ has two components, one is a three punctured sphere with boundary L_1 and one meridian from each of M_1, M_2, and the other is a disk with boundary L_2 in which two little disks inside have been taken out and Seifert surfaces for k in C_i have been attached.) Thus this filling is norm-reducing.

CLAIM 5. Let U be a surface with boundary composed of a single longitude of each of M_1 and M_2 oriented to be anti-parallel in $T - L_1$, then $\chi_-(U) \geqq 2$.

PROOF. We have to show U is not an annulus.

Let U be an annulus satisfying the conditions. Let D_1, D_2 be meridinal disks in the original torus T with G_1, G_2 on them respectively. D_1 together with D_2 separate M_1 from M_2, so clearly U has non-trivial intersection with at least one of them, say D_1. Any simple closed curve μ which belongs to $U \cap D_1$ must have G_3 and L_1 on one side and L_2 on the other, but μ is disjoint from G_2 and G_2 separates L_2 and G_3 from L_1, a contradiction.

Hence $x([U]) \geqq 2$ for any homology class represented by one of these surfaces.

Attach to P a solid torus such that the meridian of P becomes trivial. In the manifold obtained, these two longitudes bound an annulus homologous to a surface as in Claim 5 embedded in the original M. So the last filling is norm-reducing as well, and is distinct from the previous one.

So far we have two distinct norm-reducing fillings. One obtains immediately that M is irreducible, so we still have to show:

CLAIM 6. *M is Φ_P-atoroidal.*

This claim just follows by observing that the only decomposing torus which yields a cobordism (i.e. separates the other components of the boundary from P) is a boundary parallel one (in order to obtain that, one may look at meridinal disks with the curves G_1, G_2, M_1, M_2 on them and apply the same sort of argument we use in Claim 5 above).

In order to get an example with P the only boundary component, we do the following. Attach high genus knot complements to L_1, L_2 respectively such that their original longitudes are trivial homologically. Make a partial double by gluing another copy of the obtained manifold along M_1 and M_2. So far we have a manifold with two boundary components. Any filling on one side will serve as an example. In the obtained manifold the norm of homology classes represented by partial-double surfaces to the ones we use in Claims 4 and 5 is being reduced by the appropriate fillings. (Small disks are taken and minimal genus Seifert surfaces of the knot complement are added to the one of Claim 4.)

To prove (ii) assume φ is a norm-reducing filling for a homology class α which belongs to the cone over some face of K. φ induces a natural homomorphism $\varphi *: H_2(M, \partial M - P) \rightarrow H_2(M_\varphi, \partial H_\varphi)$.

Let β be any other class in the interior of the cone over the same face. Then $\beta = \mu_1\alpha + \mu_2\gamma$ where γ lies on the boundary of the face we are looking at. Denote by x_φ the Thurston norm on the manifold M_φ, clearly $x_\varphi(\varphi_*(.)) \leqq x(.)$. By knowing φ is reducing for α we get:

$$\begin{aligned} x_\varphi(\varphi_*\beta) &= x_\varphi(\varphi_*(\mu_1\alpha + \mu_2\gamma)) \\ &\leqq \mu_1 x_\varphi(\varphi_*\alpha) + \mu_2 x_\varphi(\varphi_*\gamma) < \mu_1 x(\alpha) + \mu_2 x(\gamma) = x(\beta). \end{aligned}$$

So φ is reducing for β as well. By Theorem 1, no other filling can be norm-reducing for β, so (ii) follows by uniqueness of the norm-reducing filling (for each class) and symmetry of the Thurston ball $K\,(x(\alpha)=x(-\alpha))$. Denote by Q_φ the set of classes being reduced by the Dehn filling φ. Then, if φ is not the only norm reducing filling $\partial Q_\varphi \neq \Phi$. From the above argument we know ∂Q_φ is a $(k-2)$ cell complex, where k is the rank of $H_2(M, \partial M - P)$.

If $k \leqq 1$ then $n_r(M,P) \leqq 1$, so (iii) follows.

We conclude by generalizing the construction we used, in order to obtain the following theorem:

THEOREM 7. *For any finite set of primitive slopes one can construct a 3-manifold for which the given finite set is a subset of the set of reducing fillings.*

PROOF. Let $\{(p_1,q_1),\dots,(p_n,q_n)\}$ be such a set, let T be a solid torus with boundary denoted by P and let $C_1,\dots,C_{2n}$ be concentric parallel tori in T. Let K_i, K_{n+i} be (p_i,q_i) torus knots lying on C_i, C_{n+i}, $1\leqq i\leqq n$. Let L_i be a torus knot parallel to K_i on C_i for $1\leqq i\leqq 2n$ (L_i and K_i bound an annulus on C_i). Let T' be the solid torus T after taking out regular neighborhoods of the knots $K_1, L_1,\dots,K_{2n},L_{2n}$ (Fig. 2).

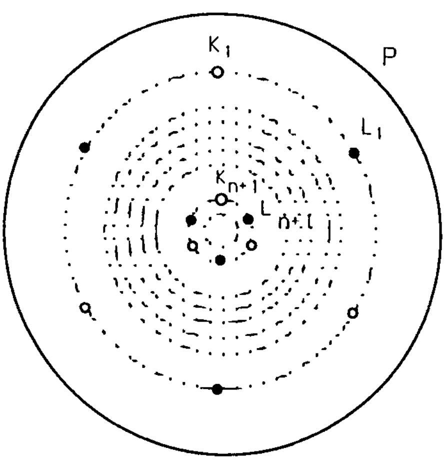

Fig. 2. A meridian disk of T'.

At this stage our aim is to modify T' so that certain minimal genus surfaces will include annuli parallel to P with the desired slopes. Such surfaces are obviously compressible under the torus fillings to P compatible with the slopes of their P-parallel annuli.

In T' we look at annuli whose boundary is composed of longitudes of K_i and L_i with anti-parallel orientation. A surface S_i homologous to that annulus containing P-parallel annulus has genus

$$\text{genus}(S_i) = 2\,\text{lk}(K_i, K_1) + \cdots + 2\,\text{lk}(K_i, K_{i-1})$$

where $\text{lk}(K_i, K_j)$ denotes the linking number between K_i and K_j under the natural imbedding of T' in S^3.

In order to make $\{S_i\}$ into minimal genus surfaces we take out regular neighborhoods of simple closed curves in T' and add high genus knot complements instead, exactly as we did in the proof of Theorem 3. For that purpose we look on the central projection from the core of T' to the torus P. This projection takes the knots K_i, L_i to simple closed curves on P such that K_i, K_{n+i} have the same image. Except for the last ambiguity one may assume the projections have only transverse double points. In order to force a surface bounded by longitudes of K_i and L_i to have at least the genus of S_i, we look at the set of these double points. If K_i (or L_i) meets K_j (or L_j) in such a double point we look at a simple closed curve surrounding K_i and K_{n+i} on a meridian disk which cuts the core of T' in a point which is close and in a clockwise direction compared to a meridian disk having the double point on it. We take out a regular neighborhood of this curve and glue back a high genus knot complement. The manifold obtained by repeating this process for all double points will be denoted by M.

In M the surface S_i has to be slightly modified by taking out disks and gluing back Seifert surfaces of the high genus knot complement, one for every high genus knot complement for which the core of its boundary torus has linking number one with either K_i or L_i (Fig. 3). One concludes immediately that the

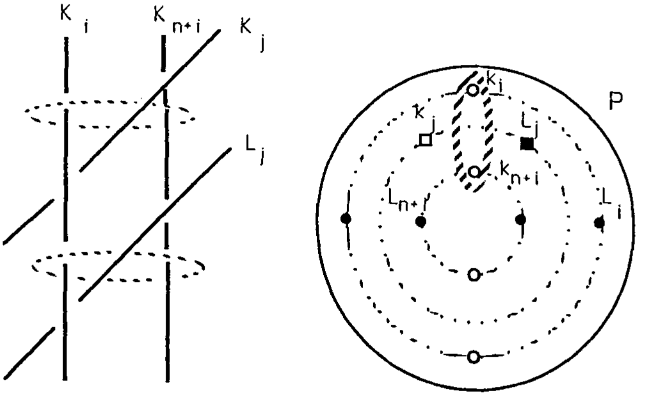

Fig. 3.

number of transverse double points between the projections of K_i and K_j (or L_j) is the linking number $\mathrm{lk}(K_i, K_j)$ between the two knots, where the knots are viewed as knots in S^3, under the natural imbedding of T' into S^3. Therefore by looking at a series of consecutive meridian disks of the original solid torus T cut by the high genus knot complements in M, one can easily show that the genus of a surface in M for which the boundary composed of longitudes of K_i and L_i with anti-parallel orientation, is bounded below by genus(S_i). Since S_i includes a P-parallel annulus, $A\,(p_i, q_i)$ Dehn filling will make it compressible.

Let N be a torus in M separating P from the other parts of ∂M. N is disjoint from K_1 and L_1 and therefore from the concentric torus C_1. Furthermore, by forming simple isotopies we can assume N is disjoint from the high genus knot complements we have added, so P and N are boundaries of a product. This shows M is Φ_P-atoroidal. If Q is a sphere in M, then Q can be modified by simple isotopies to be disjoint from the high genus knot complements and the concentric tori C_i for $1 \leqq i \leqq 2n$. Therefore Q bounds a ball B in T'. B has to be disjoint from the high genus knot complements since it lies between two consecutive concentric tori and because its boundary Q is. So B is included in M. This proves M is irreducible and we conclude that M serves as an example. By attaching high genus knot complements to K_i, L_i $(1 \leqq i \leqq 2n)$ such that their original longitudes are trivial homologically, we form an example with P the only boundary component.

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